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THE INVARIANT STRAIGHT LINES OF AN AFFINE TRANSFORMATION IN \mathbb{R}^n WITHOUT FIXED POINTS

ERWIN KASPAREK

Abstract. In this note we prove that every affine transformation in \mathbb{R}^n of the form $f(x) = Ax + a$ without fixed points has an invariant straight line if and only if when $g(a) \in \text{Im}(g \circ g)$, where $g(x) = (A - I)x$.

In [1] it was proved that an affine transformation f in \mathbb{R}^n without fixed points and which fulfils the inequality $|f(x)f(y)| \leq |xy|$ for any x and y has an invariant straight line. In this note we will solve this problem for arbitrary affine transformation in \mathbb{R}^n without fixed points.

If an affine transformation f has an invariant straight line l , then the restriction of the transformation f to l is a translation by some nonzero vector v (f has no fixed points). For a given vector $v \in \mathbb{R}^n$ the set

$$H = \{x \in \mathbb{R}^n : Ax + a = x + v\}$$

is either empty or an affine subspace of the \mathbb{R}^n . If $v = 0$ then the set H is empty because the affine transformation has no fixed points.

In this note we are going to find a nonzero vectors v such that the set H would be nonempty and invariant under the transformation f .

THEOREM 1. *An affine transformation f has an invariant affine subspace H of the \mathbb{R}^n determined by some vector $v \neq 0$ if and only if when the vector a has a decomposition $a = v + u$, where $v \in \ker g$, $u \in \text{Im } g$ and $g(x) = (A - I)x$.*

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PROOF. Necessity. If the equation

$$(1) \quad Ax + a = x + v$$

has a solution, with respect to x , for some vector v , then $v - a \in \text{Im } g$. The general solution of the system (1) has the form $x = s_1 a_1 + \dots + s_p a_p + x_0$, where the vectors a_1, \dots, a_p form a basis of the $\ker g$ and x_0 is a particular solution of the system (1). It is easy to check that $f(H) = H + v$ and thus $f(H) = H$ if and only if when $v \in \ker g$. Taking into account that $v - a \in \text{Im } g$ we obtain the assertion.

Sufficiency. If $a = v + u$, where v is a nonzero vector which belongs to $\ker g$ and u belongs to $\text{Im } g$, then the linear equation (1) can be written in the equivalent form

$$(2) \quad (A - I)x = -u.$$

Because $u \in \text{Im } g$, so $(A - I)x_1 = -u$ for some $x_1 \in \mathbb{R}^n$, thus we can rewrite the system (2) in the form

$$(3) \quad (A - I)(x - x_1) = 0.$$

Since the affine transformation f has no fixed points, we have $\det(A - I) = 0$. Thus the system (3) has a nonzero solution. The affine subspace H determined by the vector v is invariant under the transformation f , because $v \in \ker g$. \square

Theorem 1 is equivalent to the following

THEOREM 2. *An affine transformation f has an invariant affine subspace H determined by some vector v if and only if when $g(a) \in \text{Im } (g \circ g)$.*

PROOF. Necessity. From Theorem 1 we have the decomposition $a = v + u$, where $v \in \ker g$ and $u \in \text{Im } g$. Thus $g(a) = g(u)$, i.e. $g(a) \in \text{Im } (g \circ g)$.

Sufficiency. If $g(a) \in \text{Im } (g \circ g)$ then there exists some vector $w \in \mathbb{R}^n$ such that $g(a) = g(g(w))$. Thus $g(a - g(w)) = 0$. This denotes that $a - g(w) \in \ker g$. We conclude that $a = v + g(w)$ for some vector $v \in \ker g$. The vector v is nonzero because the affine transformation f has no fixed points. \square

COROLLARY 1. *If a vector a has a decomposition $a = v + u$, where v is nonzero vector belonging to $\ker g$ and u belongs to $\text{Im } g$, then the straight line l determined by the equation $x = x_0 + vt$, where x_0 is a solution of the system (1), is invariant under the affine transformation f .*

PROOF. If $x = x_0 + vt$ then $Ax + a = Ax_0 + tAv + a$. Because $v \in \ker g$, so $Av = v$ and $Ax_0 + a = x_0 + v$. Thus we obtain the equality $Ax + a = x_0 + (t+1)v$, i.e. $f(l) = l$. \square

REMARK 1. The decomposition of the vector a in the form $a = v + u$, where $v \in \ker g$ and $u \in \operatorname{Im} g$, is unique.

Indeed, let us assume that $a = v_1 + u_1$ and $a = v_2 + u_2$, then from Corollary 1 it follows that the straight lines which are specified by the equations: $x = x_0 + v_1 t$ and $x = x_0 + v_2 t$ are invariant under the affine transformation f and thus the point x_0 is the fixed point of f , and this is not possible.

COROLLARY 2. *If a linear transformation g of an affine transformation f without fixed points fulfils the equality*

$$(4) \quad \ker g = \ker(g \circ g),$$

then f has an invariant straight line.

PROOF. If $\ker g = \ker(g \circ g)$ then $\mathbb{R}^n = \ker g \oplus \operatorname{Im} g$, i.e. $a = v + u$, where $v \in \ker g$ and $u \in \operatorname{Im} g$. The vector v is nonzero because the affine transformation f has no fixed points. From Corollary 1 we obtain our assertion. \square

REMARK 2. In particular, the equality (4) is satisfied by affine transformations given by a symmetric matrix or by affine transformations such that the number 1 is a simple characteristic number of its matrix.

Reference

- [1] Kasperek E., *Remark on invariant straight lines of some affine transformations in \mathbb{R}^n without fixed points*, Ann. Math. Sil. **19** (2005), 19–21.

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